A random wave and Schell model for the mean RCS of bent chaotic ducts with a homogeneous scattered aperture field distribution.

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Abstract

It is shown how a Schell model can be constructed to estimate the mean radar cross section (RCS) of long bent engine ducts. A consistent random wave model is also provided to determine representative RCS distributions of such ducts. In both cases it is assumed that the duct has an aperture which is approximately normal to the duct axis and a scattered field distribution which is homogeneous across the aperture. It is believed that such a model is often valid for long chaotic bent ducts when most rays (in a ray representation) undergo many reflections before escaping the duct. A comparison is made with a non-convergent shooting-and-bouncing (SBR) prediction of a complex duct.

1.1 Introduction

Chaos theory [1, 2] shows that, for all but the simplest problems with high symmetry, curved ducts will amplify any error in the initial ray trajectory exponentially with the number of ray bounces. The average rate of exponential growth is characterised by the Lyapunov exponent which can be calculated directly from the compound deviation matrix, or the average ray divergence. An important consequence is that it becomes computationally impossible to accurately (in a point-wise sense) predict the RCS of even quite simple duct structures using shooting-and-bouncing (SBR) methods. In a recently submitted paper [3] it is shown how a random wave model can be used to determine representative radar cross section (RCS) distributions of a class of long straight chaotic ducts. In this paper the class of sufficiently bent ducts is considered, with important application to the design of realistic engine ducts in the aerospace industry.

Although both straight and bent ducts exhibit chaos in the high frequency limit, there are important differences in the properties of the scattered field. If a duct is both sufficiently long and sufficiently bent it will show that it is possible to apply a standard simplification of the Schell model [6] to determine the mean RCS polarisation (scattering) matrix. This follows a Lambert form, with mean RCS which is independent of the *shape* of the aperture. This is in contrast to the straight duct where it is not possible to obtain a shape-independent mean RCS.

It is also possible to describe a random wave model for sufficiently long and bent ducts, analogous to the model for straight ducts described in [3]. Numerical evidence is provided for the mutual-consistency of this model and the Lambert form. A comparison is also made using predictions from a non-convergent application of a shooting-and-bouncing (SBR) method for a suitably bent duct. Unfortunately no comparison is available with accurate RCS predictions; the nature of the problem makes this a very high complexity task.

An incident plane wave is characterised by the azimuth and elevation angles ϕ_{in} (angle made with x axis and projection of wave direction on to x-y plane) and ξ_{in} (angle made with wave and z axis). The z axis is assumed to be normal to the duct aperture and locally parallel to the duct axis near the aperture. In an SBR method the plane wave is represented by a field of ray bundles each of which is launched into the duct from the aperture with the same direction, (ϕ_{in}, ξ_{in}). The direction of a ray within the duct will be labeled by (ϕ, ξ), defined in the same manner.

The difference in the field characteristics between straight and bent ducts is related to the fact that in the high frequency limit a sufficiently long and sufficiently bent ergodic duct completely randomises the direction of the ray field, so there is complete loss of information of the direction (both ϕ_{in} and ξ_{in}) of the incident wave. On the other hand, a straight duct conserves ξ for each ray and hence the incident wave direction ξ_{in}. If the scatter angles of a scattered far field are labeled as (ϕ_s, ξ_s) then a straight duct generates an RCS distribution which is sharply peaked on a scattering cone of half angle $\xi_s \approx \xi_{in}$. The manner in which the RCS characteristics of a straight duct evolve, under perturbation, to those of a sufficiently bent one will not be examined here.

1.2 Schell models and the mean RCS of a long bent chaotic duct

In this section it is assumed that a duct is sufficiently long and bent that all rays entering the duct, under a ray representation, are completely randomised before they exit the duct. In other words the duct is ergodic with no mixed or soft chaotic effects [5]. Furthermore the coherent contribution to the scattered field from short path length rays, reflected near the aperture, is assumed negligible. This description is valid only if the duct aperture is normal to the duct axis and the duct axis bends only slowly in the neighbourhood of the aperture.

As in [3], it is necessary to define a suitable ensemble for the generation of statistics. As there, the ensemble is assumed to be the class of duct geometries whose relative dimensions do not vary significantly from the reference duct, but whose dimensional differences are large compared to a wavelength. Thus in the high frequency limit the relative changes in duct dimensions, necessary to define the ensemble, approaches zero. The *cross spectral density* at frequency ν [6], $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu)$, plays an important role in determining the mean RCS. For a harmonic wave with time dependence $e^{i\omega_0 t}$, this takes the form,

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu) = \Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)\delta(\nu - \omega_0/2\pi) \tag{1}$$

where $\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0)$ is the *mutual coherence function* at time zero. In the optical context, both cross spectral density and mutual coherence refer to fields with non-zero frequency band-width, so that ensembles are taken through observations of fields at different times. It is assumed that all averaged quantities that are defined with respect to a small but non-zero bandwidth are equal to averages constructed over the aforementioned duct ensemble. The main statistical assumption is that means defined in these different ways are equal in the high frequency limit.

The scattered field on the duct aperture is said to be *homogeneous* if the cross spectral density is independent of the absolute coordinates, i.e. that $W(\boldsymbol{r}_1, \boldsymbol{r}_2) = W(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. This is expected to be valid at high frequencies, given the earlier ray assumptions, a few wavelengths away from the duct boundary and forms the basis for the Schell model for the cross spectral density [6]. A model is first considered for a *scalar* field, exactly as in [6], with a suitable construction for the RCS. The form is then generalised to determine the mean RCS scattering matrix for the vector problem.

Under the quasi-homogeneous Schell model for a scalar field $\psi(\boldsymbol{x})$, the cross spectral density on an electrically large aperture takes the form,

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \nu) = [S^{(0)}(\boldsymbol{\rho}_1, \nu)]^{1/2}[S^{(0)}(\boldsymbol{\rho}_2, \nu)]^{1/2} g^{(0)}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \nu) \tag{2}$$

where $S^{(0)}(\boldsymbol{\rho}, \nu)$ is a slowly varying function of ρ, termed the *spectral density* and $g^{(0)}(\boldsymbol{\rho})$ is the *spectral degree of coherence*. In the harmonic limit,

$$S^{(0)}(\boldsymbol{\rho}, \nu) = S_\delta^{(0)}(\boldsymbol{\rho})\delta(\nu - \omega_0/2\pi) \tag{3}$$

and in the fully homogeneous model,

$$S_\delta^{(0)}(\boldsymbol{\rho}) = \begin{cases} S_\delta & \text{for } \boldsymbol{\rho} \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $\mathcal{A}$ represents the domain of the aperture and S_δ is a positive constant.

The un-normalised scalar field correlation function, $F_u(R)$, is defined by,

$$F_u(R) = \langle \psi(\boldsymbol{x}), \psi(\boldsymbol{x}+\boldsymbol{R}) \rangle_{\mathcal{A}} = \iint_{\mathcal{A}} \psi^\star(\boldsymbol{x})\psi(\boldsymbol{x}+\boldsymbol{R}) dx\, dy \tag{5}$$

where $R = |\boldsymbol{R}|$ represents the distance between two points on the aperture, assumed to lie on the x-y plane. The normalised correlation function may then be defined by $F(R) = F_u(R)/F_u(0)$. It is now observed that,

$$F(R) = \frac{g^{(0)}(\boldsymbol{R}, \omega_0/2\pi)}{g^{(0)}(0, \omega_0/2\pi)} \tag{6}$$

This correlation function depends on the number of angular degrees of freedom in the permitted directions of a random set of waves. For a straight duct, it can be shown that

$$F(R) = J_0(k_0 R \sin \xi_{in}) \qquad \text{[ergodic straight duct]} \tag{7}$$

For a sufficiently bent duct, a random scalar wave expansion implies,

$$F(R) = \frac{\sin(k_0 R)}{k_0 R} \qquad \text{[ergodic and sufficiently bent duct]} \tag{8}$$

The latter result may be derived under the assumption that the wave field is composed of a random sum of plane waves moving in arbitrary directions. (See e.g. [4] for a general formula in n-dimensional space). An explicit representation is given for vector fields in the next section.

It is shown in [6] that the *radiant intensity* $J(\boldsymbol{s}, \nu)$ in the scattering direction $\boldsymbol{s}$ is given by,

$$J(\boldsymbol{s}, \nu) = (2\pi k)^2 \tilde{W}^{(0)}(-k\boldsymbol{s}_\perp, k\boldsymbol{s}_\perp, \nu) \cos^2 \xi_s \tag{9}$$

where $k = 2\pi\nu/C$ and $\tilde{W}^{(0)}$ represents the double Fourier transform of $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \nu)$ with respect to $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. In the harmonic limit,

$$J(\boldsymbol{s}, \nu) = J_\delta(\boldsymbol{s}, \boldsymbol{s}_{in}) \delta(\nu - \omega_0/2\pi) \tag{10}$$

where the dependence on the direction $\boldsymbol{s}_{in}$ of the incident plane wave is made explicit. This allows an identification of the mean bistatic RCS, $\overline{\sigma}$, given by

$$\overline{\sigma} = \frac{4\pi J_\delta(\boldsymbol{s}, \boldsymbol{s}_{in})}{P_0(\boldsymbol{s}_{in})} \tag{11}$$

where $P_0(\boldsymbol{s}_{in})$ is an as yet unspecified positive function of $\boldsymbol{s}_{in}$. Using (2),

$$\tilde{W}^{(0)}(\boldsymbol{f}_1, \boldsymbol{f}_2, \nu) = \frac{1}{(2\pi)^4} \int_\infty \int_\infty \sqrt{S^{(0)}(\boldsymbol{\rho}_1, \nu) S^{(0)}(\boldsymbol{\rho}_2, \nu)}\, g^{(0)}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2, \nu) e^{i(\boldsymbol{f}_1 \cdot \boldsymbol{\rho}_1 + \boldsymbol{f}_2 \cdot \boldsymbol{\rho}_2)} d^2\rho_1 d^2\rho_2 \tag{12}$$

Crucially, provided that $g^{(0)}(\boldsymbol{\rho}, \nu)$ decays sufficiently rapidly with $|\boldsymbol{\rho}|$, for large $|\boldsymbol{\rho}|$, then

$$J(\boldsymbol{s}, \nu) \approx (2\pi k)^2 \tilde{S}^{(0)}(0, \nu) \tilde{g}^{(0)}(k\boldsymbol{s}_\perp, \nu) \cos^2 \xi_s \tag{13}$$

where

$$\tilde{S}^{(0)}(\boldsymbol{f},\nu) = \frac{1}{(2\pi)^2}\int_{\infty} S^{(0)}(\boldsymbol{\rho},\nu)e^{i\boldsymbol{f}\cdot\boldsymbol{\rho}}d^2\rho \tag{14}$$

and

$$\tilde{g}^{(0)}(\boldsymbol{f}',\nu) = \frac{1}{(2\pi)^2}\int_{\infty} g^{(0)}(\boldsymbol{\rho}',\nu)e^{i\boldsymbol{f}'\cdot\boldsymbol{\rho}'}d^2\rho' \tag{15}$$

The integration over ∞ refers to integration over the entire plane containing the aperture, ξ_s is the scatter angle made with the z-axis (perpendicular to the aperture). The direction $\boldsymbol{s}_\perp$ is the transverse component of the scattering vector, $\boldsymbol{s}$, in the x-y plane, with magnitude $\sin\xi_s$. (Note the $+i$, rather than the $-i$ used in Mandel and Wolf [6], consistent with the harmonic convention used previously). Using (6) it is observed that the form (13) is permitted for the bent duct but *not* for the straight duct because a zero order Bessel function decays insufficiently rapidly for large argument.

Using (11) together with (13), (6), (8) and (15), an expression for the mean RCS may be obtained, assuming a scalar field, of the form,

$$\overline{\sigma} = f(\xi_{in},\phi_{in})\cos\xi_s \tag{16}$$

for some as yet unspecified positive valued function $f(\xi_{in},\phi_{in})$ independent of scatter angles. However, under reciprocity, $\overline{\sigma}$ should be invariant under interchange of transmitter and receiver so that,

$$\overline{\sigma} = \beta\cos\xi_{in}\cos\xi_s \tag{17}$$

for an as yet unspecified positive constant β. If the duct is perfectly conducting and if the projected area of the aperture is large enough, such that $k_0L\cos\xi_{in} \gg 1$ and $k_0L\cos\xi_s \gg 1$ where L is a measure of the aperture size (such as the radius of the maximum inscribed circle within the aperture), then conservation of power implies,

$$\overline{\sigma} = 4A_s\cos\xi_{in}\cos\xi_s \tag{18}$$

where A_s is the area of the duct aperture. This is a version of *Lambert's law* [6]. Interestingly, (18) can also be derived without use of the correlation function $F(R)$. If it is assumed that, (1) the distribution of scattered power is independent (up to a constant factor) of the direction of the incident wave, (2) the mean RCS is reciprocal and (3) the intercepted power is proportional to the projected area of the duct, then (18) follows. Assumptions (2) and (3) are true of a straight duct but (1) is not.

It is possible to generalise the scalar result to vector fields. It is necessary to make the above assumptions and, in addition, assume that the scattered field normal to the aperture has no preferred polarisation. Using reciprocity arguments on the bistatic scattering matrix [7] it is then possible to show that the co- and cross-polar terms of the mean RCS are given by,

$$\begin{pmatrix} \overline{\sigma}_{l'l} & \overline{\sigma}_{l'm} \\ \overline{\sigma}_{ml} & \overline{\sigma}_{mm} \end{pmatrix} = 2A_s\cos\xi_s\cos\xi_{in}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{19}$$

with the polarisation base defined with respect to the scattering plane containing the receiver, transmitter and scatterer. The notation $\overline{\sigma}_{pq}$ refers to the mean RCS as observed by a receiver polarised in the direction $\boldsymbol{p}$ for a transmitter polarised in the direction $\boldsymbol{q}$. The $\boldsymbol{m}$ direction lies orthogonal to the scattering plane. The $\boldsymbol{l}'$ direction is orthogonal to $\boldsymbol{m}$ and the direction of the scattered wave. The $\boldsymbol{l}$ direction is orthogonal to $\boldsymbol{m}$ and the direction of the incident wave. This result has a simple interpretation, namely that the mean RCS is independent of the polarisation of the incident plane wave and of the polarisation of the receiver. Power is uniformly scattered into both co and cross-polar directions, hence the factor of 2 rather than 4 in (19).

1.3 A random wave description for very bent ducts

It is possible to represent the field across the duct aperture of a sufficiently bent duct by a sum over randomly directed plane waves, much as in [3] for straight ducts. Here only a fixed incidence angle is considered, though it is believed that the method described in [3] can be generalised to achieve suitable angle of incidence variation. The random wave model is implicit in the functional form for the correlation function (8). Just as in [3] it is possible to represent the total aperture field as the sum of a random part and a non-random part, where the non-random part is evaluated as the sum of a convergent ray field using a shooting and bouncing (SBR) or related method. The random transverse field component on the aperture may then be written as,

$$\boldsymbol{E}_{\perp}(\boldsymbol{x}_{\perp}) = A_2 \sum_{l=1}^{M} \boldsymbol{p}_{\perp l} w_l e^{-j[k_0 \sin \xi_l (x \cos \alpha_l + y \sin \alpha_l) + \beta_l]} \qquad \text{for } (x,y) \in \mathcal{A} \tag{20}$$

where

$$A_2 = \sqrt{\frac{N_B}{M N_{tot}}} \tag{21}$$

N_B represents the number of non-convergent rays to be represented by the random field and N_{tot} is the total number of launched rays in the SBR formulation. The quantities α_l and β_l are uncorrelated random variables uniformly distributed over the interval $[0, 2\pi]$. Similarly, ξ_l are uncorrelated random variables distributed over the interval $[0, \pi/2]$. $\boldsymbol{p}_{\perp l}$ represents the transverse component of any possible polarisation associated with the l_{th} wave direction. This is given by,

$$\boldsymbol{p}_{\perp l} = (\cos \xi_l \cos \alpha_l \cos \eta_l + \sin \alpha_l \sin \eta_l)\, \hat{\boldsymbol{x}} + (\cos \xi_l \sin \alpha_l \cos \eta_l - \cos \alpha_l \sin \eta_l)\, \hat{\boldsymbol{y}} \tag{22}$$

where η_l is an uncorrelated random variable distributed over $[0, 2\pi]$. The term w_l represents a weight function which can be argued is independent of all random variables except ξ_l. Similar to the definition on a scalar field, the scalar correlation function on a vector field is given by,

$$F_u(R, u) = \int\!\!\int_{\mathcal{A}} \boldsymbol{E}^{\star}(\boldsymbol{x}'_{\perp}).\boldsymbol{E}(\boldsymbol{x}'_{\perp} + \boldsymbol{R}) dx\, dy \tag{23}$$

where $\boldsymbol{R} = R\hat{\boldsymbol{x}} \cos u + R\hat{\boldsymbol{y}} \sin u$. If the field is assumed to be totally random, i.e. $N_B = N_{tot}$, then (20) implies that for large M,

$$F_u(R, u) \approx \frac{A_s}{M} \sum_{l=1}^{M} |w_l(\xi_l)|^2 e^{-jk_0 R \sin \xi_l (\cos u \cos \alpha_l + \sin u \sin \alpha_l)} \tag{24}$$

Application of the Monte-Carlo integration theorem (e.g. [10]) to the sum (24) over random variables for large M replaces the sum by an integral and reduces to,

$$F_u(R) \approx \frac{2A_s}{\pi} \int_0^{\pi/2} w^2(\xi)\, J_0(k_0 r \sin \xi)\, d\xi \tag{25}$$

where $J_0(z)$ is the zero order Bessel function and $w(\xi)$ is defined as a positive real function. Assuming that $F_u(R)$ takes the form (8), this requires that

$$w(\xi) = \sqrt{|u_0 \sin \xi|} \tag{26}$$

Using conservation of energy arguments for a large aperture duct it can be shown that,

$$u_0 = \pi |\cos \xi_{in}| \tag{27}$$

In other words,

$$w_l(\xi_l) = \sqrt{\pi |\cos \xi_{in} \sin \xi_l|} \tag{28}$$

Finally, an estimate is required for a finite lower bound to M. As a rough measure, if the area of the aperture is A_s, then the number of square wavelengths is A_s/λ^2 where $\lambda = 2\pi/k_0$ is the free space wavelength. According to the two dimensional Nyquist criterion, each square wavelength must be sampled at least 4 times to resolve the information. This implies a lower limit on M set by $M > k_0^2 A_s/\pi^2$. Because of the random non-uniform sampling it is expected that $M \gg k_0^2 A_s/\pi^2$ and in the example below a much larger figure is assumed.

With the transverse scattered aperture field defined, the RCS may be determined using a standard near-to-far field transformation, e.g. as described in [9].

1.4 Some comparisons and the use of SBR methods

As already stated, it is impossible to achieve numerical convergence with respect to ray bundle density for a long chaotic duct. It is common (if dangerous) practise to use such methods non-convergently, in which case predictions are rather algorithm dependent. SBR methods often "relegate" first order ray bundles to zero order bundles of a constant predefined bundle width if excessive divergence occurs. For long chaotic ducts almost all ray bundles will be relegated in this manner. If the predefined bundle width is large compared to a wavelength one might expect some degree of similarity with the above random wave model, assuming aperture field correlations can be ignored for R greater than the bundle width.

Some results are provided for a long complex duct similar to one in a real aircraft designed so that the end of the duct is not visible at the aperture. The aperture is chosen to be approximately rhomboid with side length L_p and included angle of 60^o, such that $k_0 L_p \approx 190$ and normal to the duct axis. A relegated ray bundle is assumed to have a three wavelength square cross section. In these predictions a normal incidence plane wave is assumed and the predicted bistatic RCS is shown along an azimuth cut. Predictions are made at 15GHz with $L_p \approx 0.60$m. Figures 1 to 4 illustrate the SBR predictions, for each of the four terms of the scattering matrix. The predicted Lambert form is also illustrated and the agreement is good for these examples, though this is not always so. For example it has been found that non-convergent SBR predictions do not always provide similar levels of agreement for plane waves far from normal incidence (but not so far that $k_0 L \cos \xi_{in} \not\gg 1$). In this case, however, there is evidence to suggest that the SBR predictions of mean RCS may be dependent on algorithm-specific details so it is not possible to come to any strong conclusions here.

In figures 5 and 6 the random wave model is applied, assuming an aperture defined as a perfect 60^o rhombus with side length $L_p \approx 2.187$m and $k_0 = 80.0\text{m}^{-1}$, with one side parallel to the x-axis. Once again a normal incidence plane wave is assumed and the bistatic RCS is shown along a cut parallel to the y-axis for the two receiver polarisations (with scattered E-field in the direction of increasing ξ_s and increasing ϕ_s). Three members of the ensemble are illustrated, each with $M = 83529$, approximately 31 times the Nyquist limit. Although it is not possible to come to strong conclusions based only on these results, several other aperture geometries have also been studied. Performing an ensemble average, or an average over ϕ_s, has always shown good agreement with the Lambert form away from grazing angles (for $k_0 L \cos\xi_s \gg 1$ and $k_0 L \cos\xi_{in} \gg 1$).

1.5 Conclusions

The Lambert form for the mean RCS and the random wave models appear to be in agreement. Both appear to be good models for long, slowly bending chaotic ducts when the aperture is normal to the duct axis, as evidenced by a non-convergent use of an SBR method. However, it is likely that non-convergent use of an SBR method will not always be statistically accurate and there is a pressing need for verifiably accurate predictions or quality measurements of complex duct structures.

Finally, it should be mentioned that it is possible to extend these ideas to study ducts with apertures that are not transverse to the duct axis. Provided there exists a point in the duct not too far from the aperture where the fields can be described as random across a cross section, the random fields at this cross section can be mapped to the aperture. One such method is to employ the *generalised radiance function* [6] to locally describe a random field. A density mapping on a surface of section in phase space can be used to map this to the aperture. It is hoped to describe this elsewhere [11].

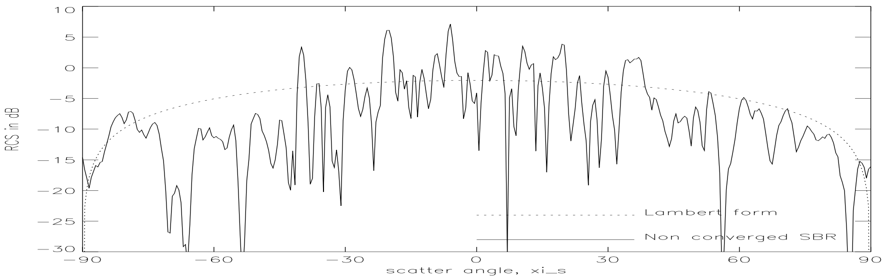

Figure 1. *VV polarisation, (horizontal cut, bistatic)*

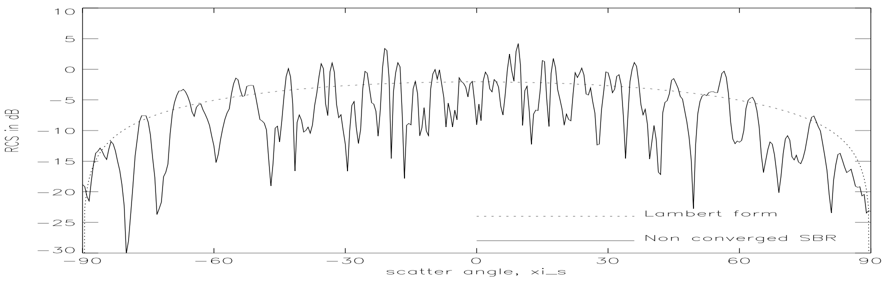

Figure 2. *VH polarisation, (horizontal cut, bistatic)*

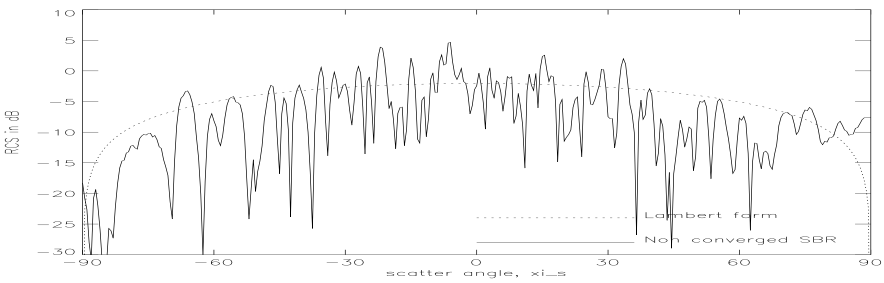

Figure 3. *HV polarisation, (horizontal cut, bistatic)*

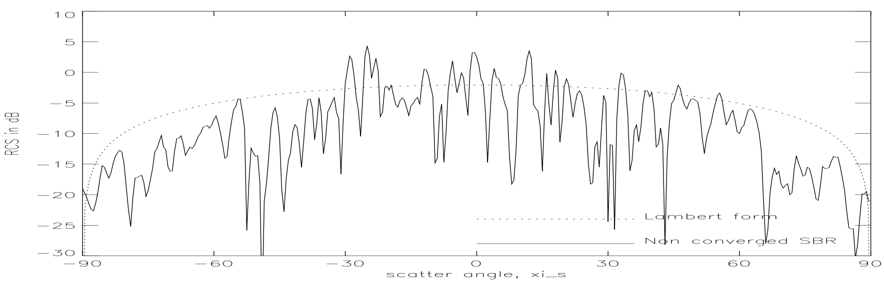

Figure 4. *HH polarisation, (horizontal cut, bistatic)*

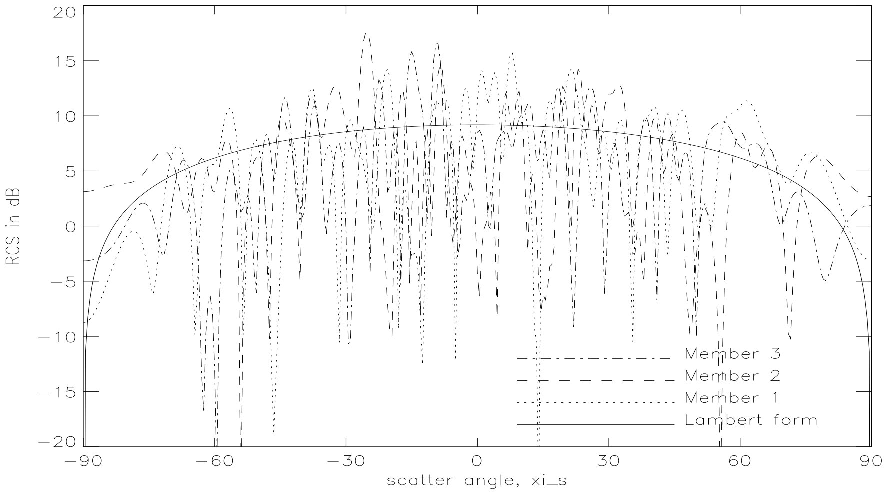

Figure 5. *Random wave RCS, $\xi-$polarised receiver*

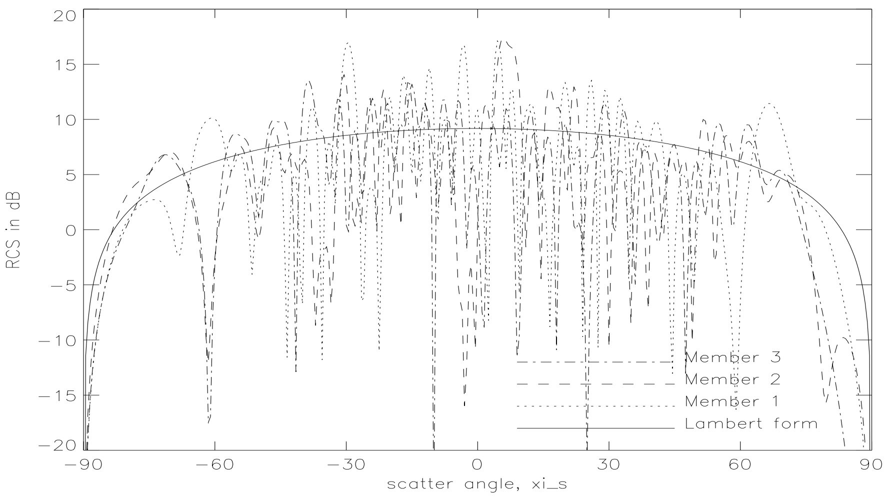

Figure 6. *Random wave RCS, $\phi-$polarised receiver*

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